DETERMINING VELOCITIES OF BODIES MOVING IN A VISCOUS LIQUID UNDER THE ACTION OF ELECTROMAGNETIC FORCES

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The model problem of induction into motion of a sphere located in a conductive liquid by the electromagnetic field induced in that liquid by a magnetic dipole $\mathbf{m} = \mathbf{m}_0 e^{i\omega t}$ offset relative to the center of the sphere was considered in [1]. The analysis was carried out in the Stokes approximation, with the study of the effect upon the liquid limited to only that portion of the force field responsible for setting the sphere in motion, namely

$$\mathbf{f} = \frac{45}{4\pi}\varepsilon\frac{H_0^2}{\delta}\left(\frac{a}{r}\right)^2 e^{-2(r-a)/\delta}\sin^2\vartheta\cos\vartheta\,\mathbf{e}_r. \tag{1}$$

Here $\varepsilon = d/a$; d is the distance of the dipole $\mathbf{m}$ from the center of the sphere; a is the sphere radius; $\delta = \sqrt{c^2/2\pi\sigma\omega}$ is the skin layer thickness; σ is the liquid conductivity; $H_0 = m_0/a^3$ is the characteristic magnetic field intensity; we use a spherical coordinate system (r, ϑ, φ).

Direct solution (a technically cumbersome task) of the Stokes equation will yield the velocity of sphere motion

$$U_\infty \simeq \varepsilon\left(\frac{\delta}{a}\right)^2\frac{3aH_0^2}{8\pi\rho\nu}. \tag{2}$$

The present study will obtain a more general relationship for determining the velocity of a body set in motion by volume electromagnetic forces distributed outside the body.

The relationships obtained are valid under the following conditions: 1) Reynolds number Re $\ll$ 1 (Stokes flow); 2) volume force distribution independent of velocity field, which upon satisfaction of Eq. (1) is possible if $E_0 \gg U_0H_0/c$ (E_0, H_0 being the characteristic electric and magnetic field intensities, U_0, the characteristic flow velocity).

The system of equations describing steady state motion of the liquid in a coordinate system fixed to the body has the form

$$(\mathbf{u}\nabla)\mathbf{u} + (1/\rho)\nabla p = (1/\rho)\mathbf{f} + \nu\Delta\mathbf{u},\ \operatorname{div}\mathbf{u} = 0. \tag{3}$$

On the body surface the adhesion condition

$$\mathbf{u}|_S = 0 \tag{4}$$

must be satisfied. On a surface infinitely removed from the body one usually specifies the incident flow velocity $\mathbf{u}|_{|\mathbf{r}|\to\infty} = -\mathbf{U}_\infty$ (where $\mathbf{U}_\infty$ is the velocity vector of the body motion). However in our case the value of $\mathbf{U}_\infty$ is unknown and is defined by the condition of equality to zero of the net force acting on the body in steady state motion in the viscous liquid:

$$\int_G \mathbf{f}\,d^3\mathbf{r} = \int_S T(\mathbf{u})\,\mathbf{n}\,dS. \tag{5}$$

Here G is the region occupied by the liquid; S is the surface of the body flowed over; n is the external normal to the surface S; $T(\mathbf{u})$ is the stress tensor $T_{ij}(\mathbf{u}) = -p\delta_{ij} + \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$.

In the Stokes approximation Eq. (3) can be written as

$$\operatorname{div} T(\mathbf{u}) + \mathbf{f} = 0. \tag{6}$$

Integrating Eq. (6) over the volume G occupied by the liquid, and employing the Ostrogradskii–Gauss formula, we obtain

$$-\int_S T(\mathbf{u})\,\mathbf{n}\,dS + \int_{\Sigma_R} T(\mathbf{u})\,\mathbf{n}\,dS + \int_G \mathbf{f}\,d^3\mathbf{r} = 0 \tag{7}$$

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(Σ_R is an arbitrary surface removed from the body). Comparing Eqs. (5) and (7), we find

$$\int_{\Sigma_R} T(\mathbf{u})\,\mathbf{n}\,dS = 0. \tag{8}$$

In the case of conventional Stokes flow, where volume forces are absent ($\mathbf{f} \equiv 0$), from Eq. (7) we find

$$\int_{\Sigma_R} T(\mathbf{v})\,\mathbf{n}\,dS = \int_S T(\mathbf{v})\,\mathbf{n}\,dS. \tag{9}$$

We now use Green's formula

$$\int_G \mathbf{v}(\mu\Delta\mathbf{u} - \nabla p)\,d^3\mathbf{r} - \int_G \mathbf{u}(\mu\Delta\mathbf{v} - \nabla q)\,d^3\mathbf{r} = \int_\Sigma \{\mathbf{v}T(\mathbf{u})\,\mathbf{n} - \mathbf{u}T(\mathbf{v})\,\mathbf{n}\}\,dS, \tag{10}$$

where $\mathbf{u}$, $\mathbf{v}$ are arbitrary smooth solenoidal vectors; p, q are arbitrary smooth functions; $T(\mathbf{u})$, $T(\mathbf{v})$ are stress tensors corresponding to the fields $(\mathbf{u}, p)$ and $(\mathbf{v}, q)$.

We choose as $\mathbf{u}$, p the solution of the problem of motion of the solid under consideration under the action of electromagnetic forces [the problem of Eqs. (6), (4), (5)], and as $\mathbf{v}$, q, the solution of the conventional Stokes problem of flow over the same body. Then with consideration of Eq. (6) and boundary conditions (4) we write Eq. (10) in the form

$$-\int_G \mathbf{v}\mathbf{f}\,d^3\mathbf{r} = \int_{\Sigma_R} \mathbf{v}T(\mathbf{u})\,\mathbf{n}\,dS - \int_{\Sigma_R} \mathbf{u}T(\mathbf{v})\,\mathbf{n}\,dS. \tag{11}$$

As $|\mathbf{r}| \to \infty$ let

$$\mathbf{u} \to (U_\infty, 0, 0),\ \mathbf{v} \to (V_\infty, 0, 0). \tag{12}$$

Then on the surface Σ_R enclosing the body we have

$$\int_{\Sigma_R} \mathbf{v}T(\mathbf{u})\,\mathbf{n}\,dS = V_\infty \int_{\Sigma_R} T_{x_j}(\mathbf{u})\,n_j\,dS + \int_{\Sigma_R} O(\mathbf{v} - V_\infty\mathbf{i})\,T(\mathbf{u})\,\mathbf{n}\,dS,$$

$$\int_{\Sigma_R} \mathbf{u}T(\mathbf{v})\,\mathbf{n}\,dS = U_\infty \int_{\Sigma_R} T_{x_j}(\mathbf{v})\,n_j\,dS + \int_{\Sigma_R} O(\mathbf{u} - U_\infty\mathbf{i})\,T(\mathbf{v})\,\mathbf{n}\,dS.$$

Letting $R \to \infty$, in light of Eqs. (8), (9), (12) and the arbitrary nature of the surface we obtain

$$\int_{\Sigma_{R\to\infty}} \mathbf{v}T(\mathbf{u})\,\mathbf{n}\,dS = 0, \quad \int_{\Sigma_{R\to\infty}} \mathbf{u}T(\mathbf{v})\,\mathbf{n}\,dS = U_\infty \int_S T_{x_j}(\mathbf{v})\,n_j\,dS. \tag{13}$$

From Eqs. (11) and (13) we find the flow velocity at infinity in the case of motion of the solid body under the action of electromagnetic forces

$$U_\infty = \int_G \mathbf{v}\mathbf{f}\,d^3\mathbf{r} \Big/ \int_S T_{x_j}(\mathbf{v})\,n_j\,dS. \tag{14}$$

Equation (14) solves the problem posed. We will apply Eq. (14) to the special case of motion of a sphere under the action of an electromagnetic force $\mathbf{f}$ [Eq. (1)]:

$$\int_S T_{x_j}(\mathbf{v})\,n_j\,dS = 6\pi\rho\nu V_\infty a,$$
$$v_r = V_\infty \cos\vartheta\left[1 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\left(\frac{a}{r}\right)^3\right],\quad v_\vartheta = -V_\infty \sin\vartheta\left[1 - \frac{3}{4}\frac{a}{r} - \frac{1}{4}\left(\frac{a}{r}\right)^3\right]. \tag{15}$$

For conventional Stokes flow Eq. (15) can be found, for example, in [2]. Substituting Eqs. (1), and (15) in Eq. (14), we obtain the sphere velocity U_∞, defined by Eq. (2).

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LITERATURE CITED

1. V. I. Khonichev and V. I. Yakovlev, "Motion of a sphere in a infinite conductive liquid, produced by a variable magnetic field located within a sphere," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1978).
2. L. D. Landau and E. M. Lifshits, Hydrodynamics [in Russian], Nauka, Moscow (1988).